

CONVEXITY AND A SUM-PRODUCT TYPE ESTIMATE

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ABSTRACT. In this paper we further study the relationship between convexity and additive growth, building on the work of Schoen and Shkredov ([5]) to get some improvements to earlier results of Elekes, Nathanson and Ruzsa ([1]). In particular, we show that for any finite set $A \subset \mathbb{R}$ and any strictly convex or concave function f ,

$$|A + f(A)| \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}}$$

and

$$\max\{|A - A|, |f(A) + f(A)|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{2/11}}.$$

For the latter of these inequalities, we go on to consider the consequences for a sum-product type problem.

1. INTRODUCTION

Given a finite set $A \subset \mathbb{R}$, the elements of A can be labeled in ascending order, so that

$$a_1 < a_2 < \cdots < a_n.$$

A is said to be *convex*, if

$$a_i - a_{i-1} < a_{i+1} - a_i,$$

for all $2 \leq i \leq n-1$, and it was proved by Elekes, Nathanson and Ruzsa ([1]) that $|A \pm A| \geq |A|^{3/2}$, an estimate which stood as the best known for a decade, under various guises. Schoen and Shkredov ([5]) recently made significant progress by proving that for any convex set A ,

$$|A - A| \gg \frac{|A|^{8/5}}{(\log |A|)^{2/5}},$$

and

$$|A + A| \gg \frac{|A|^{14/9}}{(\log |A|)^{2/3}}.$$

See [5] and the references contained within for more details on this problem and its history.

2000 *Mathematics Subject Classification.* 11B75.

Key words and phrases. sumset, difference set, productset, convexity, additive energy.

In [1], a number of other results were proved connecting convexity with large sumsets. In particular, it was shown that, for any convex or concave function f and any finite set $A \subset \mathbb{R}$,

$$(1) \quad \max\{|A + A|, |f(A) + f(A)|\} \gg |A|^{5/4},$$

and

$$(2) \quad |A + f(A)| \gg |A|^{5/4}.$$

By choosing particularly interesting convex or concave functions f , these results immediately yield interesting corollaries. For example, if we choose $f(x) = \log x$, then (1) immediately yields a sum-product estimate. Furthermore, if $f(x) = 1/x$, then (2) gives information about another problem posed by Erdős and Szemerédi ([2]).

In this paper, the methods used by Schoen and Shkredov ([5]) are developed further in order to improve on some other results from [1]. In particular, the bounds in (1) and (2) are improved slightly, in the form of the following results.

Theorem 1.1. *Let f be any continuous, strictly convex or concave function on the reals, and $A, C \subset \mathbb{R}$ be any finite sets such that $|A| \approx |C|$. Then*

$$|f(A) + C|^6 |A - A|^5 \gg \frac{|A|^{14}}{(\log |A|)^2}.$$

In particular, choosing $C = f(A)$, this implies that

$$\max\{|f(A) + f(A)|, |A - A|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{2/11}}.$$

Theorem 1.2. *Let f be any continuous, strictly convex or concave function on the reals, and $A, C \subset \mathbb{R}$ be any finite sets such that $|A| \approx |C|$. Then*

$$|f(A) + C|^{10} |A + A|^9 \gg \frac{|A|^{24}}{(\log |A|)^2}.$$

In particular, choosing $C = f(A)$, this implies that

$$\max\{|f(A) + f(A)|, |A + A|\} \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}}.$$

Theorem 1.3. *Let f be any continuous, strictly convex or concave function on the reals, and $A \subset \mathbb{R}$ be any finite set. Then*

$$|A + f(A)| \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}}.$$

Applications to sum-product estimates. By choosing $f(x) = \log x$ and applying Theorems 1.1 and 1.2, some interesting sum-product type results can be specified, especially in the case when the productset is small. A sum-product estimate is a bound on $\max\{|A + A|, |A \cdot A|\}$, and it is conjectured that at least one of these sets should grow to a near maximal size. Solymosi ([7]) proved that

$\max\{|A + A|, |A \cdot A|\} \gg \frac{|A|^{4/3}}{(\log |A|)^{1/3}}$, and this is the current best known bound. See [7] and the references contained therein for more details on this problem and its history.

In a similar spirit, one may conjecture that at least one of $|A - A|$ and $|A \cdot A|$ must be large, and indeed this is somewhat true. In an earlier paper of Solymosi ([6]) on sum-product estimates, it was proved that

$$\max\{|A + A|, |A \cdot A|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{3/11}}.$$

It is easy to change the proof slightly in order to replace $|A + A|$ with $|A - A|$ in the above, however, in Solymosi's subsequent paper on sum-product estimates, this was not the case. So, $\max\{|A - A|, |A \cdot A|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{3/11}}$ represents the current best known bound of this type. Applying Theorem 1.1 with $f(x) = \log x$, and noting that $|f(A) + f(A)| = |A \cdot A|$, we get the following very marginal improvement.

Corollary 1.4.

$$(3) \quad |A \cdot A|^6 |A - A|^5 \gg \frac{|A|^{14}}{(\log |A|)^2}.$$

In particular, this implies that

$$\max\{|A \cdot A|, |A - A|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{2/11}}.$$

By applying Theorem 1.2 in the same way, we establish that

$$(4) \quad |A \cdot A|^{10} |A + A|^9 \gg \frac{|A|^{24}}{(\log |A|)^2}.$$

In the case when the product set is small, then (3) and (4) show that the sumset and difference set grow non-trivially. This was shown in [3], and here we get a more explicit version of the same result.

2. NOTATION AND PRELIMINARIES

Throughout this paper, the symbols \ll , \gg and \approx are used to suppress constants. For example, $X \ll Y$ means that there exists some absolute constant C such that $X < CY$. $X \approx Y$ means that $X \ll Y$ and $Y \ll X$. Also, all logarithms are to base 2.

For sets A and B , let $E(A, B)$ be the additive energy of A and B , defined in the usual way. So, defining $\delta_{A,B}(s)$ (and respectively $\sigma_{A,B}(s)$) to be the number of representations of an element s of $A - B$ (respectively $A + B$), and $\delta_A(s) = \delta_{A,A}(s)$, we define

$$E(A, B) = \sum_s \delta_A(s) \delta_B(s) = \sum_s \delta_{A,B}(s)^2 = \sum_s \sigma_{A,B}(s)^2.$$

Given a set $A \subset \mathbb{R}$ and some $s \in \mathbb{R}$, define $A_s := A \cap (A + s)$. A crucial observation to make is that $|A_s| = \delta_A(s)$. In this paper, following [5], the third moment energy $E_3(A)$ will also be studied, where

$$E_3(A) = \sum_s \delta_A(s)^3.$$

In much the same way, we define

$$E_{1.5}(A) = \sum_s \delta_A(s)^{1.5}.$$

Later on, we will need the following lemma, which was proven in [3].

Lemma 2.1. *Let A, B be any sets. Then*

$$E_{1.5}(A)^2 \cdot |B|^2 \leq E_3(A)^{2/3} \cdot E_3(B)^{1/3} \cdot E(A, A + B).$$

3. SOME CONSEQUENCES OF THE SZEMERÉDI-TROTTER THEOREM

The main preliminary result is an upper bound on the number of high multiplicity elements of a sumset, a result which comes from an application of the Szemerédi-Trotter incidence theorem ([8]).

Theorem 3.1. *Let \mathcal{P} be a set of points in the plane and \mathcal{L} a set of curves such that any pair of curves intersect at most once. Then,*

$$|\{(p, l) \in \mathcal{P} \times \mathcal{L} : p \in l\}| \leq 4(|\mathcal{P}||\mathcal{L}|)^{2/3} + 4|\mathcal{P}| + |\mathcal{L}|.$$

Remark. While this paper was in the process of being drafted, a very similar result to the following lemma was included in a paper of Schoen and Shkredov ([4], Lemma 24) which was posted on the arXiv. See their paper for an alternative description of this result and proof. Note also that a weaker version of this result was also proven in [3].

Lemma 3.2. *Let f be a continuous, strictly convex or concave function on the reals, and $A, B, C \subset \mathbb{R}$ be finite sets such that $|B||C| \gg |A|^2$. Then for all $\tau \geq 1$,*

$$(5) \quad |\{x : \sigma_{f(A), C}(x) \geq \tau\}| \ll \frac{|A + B|^2 |C|^2}{|B| \tau^3},$$

and

$$(6) \quad |\{y : \sigma_{A, B}(y) \geq \tau\}| \ll \frac{|f(A) + C|^2 |B|^2}{|C| \tau^3}.$$

Proof. Let $G(f)$ denote the graph of f in the plane. For any $(\alpha, \beta) \in \mathbb{R}^2$, put $L_{\alpha, \beta} = G(f) + (\alpha, \beta)$. Define a set of points $\mathcal{P} = (A + B) \times (f(A) + C)$, and a set of curves $\mathcal{L} = \{L_{b, c} : (b, c) \in B \times C\}$. By convexity or concavity, $|\mathcal{L}| = |B||C|$, and any pair of curves from \mathcal{L} intersect at most once. Let \mathcal{P}_τ be the set of points of \mathcal{P} belonging to at least τ curves from \mathcal{L} . Applying the aforementioned Szemerédi-Trotter theorem to \mathcal{P}_τ and \mathcal{L} ,

$$\tau |\mathcal{P}_\tau| \leq 4(|\mathcal{P}_\tau||B||C|)^{2/3} + 4|\mathcal{P}_\tau| + |B||C|.$$

Now we claim for any $\tau > 0$ one has

$$(7) \quad |\mathcal{P}_\tau| \ll \frac{|B|^2|C|^2}{\tau^3}.$$

The reason is as follows. Firstly, since there is no point of \mathcal{P} belonging to at least $\min\{|B|+1, |C|+1\}$ curves from \mathcal{L} , to prove (7) we may assume that $\tau \leq \sqrt{|B||C|}$. Secondly, if $\tau < 8$, then (7) holds true since

$$|\mathcal{P}_\tau| \leq |\mathcal{P}| = |(A+B) \times (f(A)+C)| \leq |A|^2|B||C| \ll |B|^2|C|^2 \leq 64 \frac{|B|^2|C|^2}{\tau^2}.$$

Finally, we may assume that $8 \leq \tau \leq \sqrt{|B||C|}$. In this case we have

$$\frac{\tau|\mathcal{P}_\tau|}{2} \leq 4(|\mathcal{P}_\tau||B||C|)^{2/3} + |B||C|.$$

Thus

$$|\mathcal{P}_\tau| \ll \max\left\{\frac{|B|^2|C|^2}{\tau^3}, \frac{|B||C|}{\tau}\right\} = \frac{|B|^2|C|^2}{\tau^3}.$$

This proves the claim (7).

Next, suppose $\sigma_{f(A),C}(x) \geq \tau$. There exist τ distinct elements $\{a_i\}_{i=1}^\tau$ from A , τ distinct elements $\{c_i\}_{i=1}^\tau$ from C , such that $x = f(a_i) + c_i$ ($\forall i$). Now we define $B_i \triangleq a_i + B$ ($\forall i$) and $\mathcal{M}_x(s) \triangleq \sum_{i=1}^\tau \chi_{B_i}(s)$, where $\chi_{B_i}(\cdot)$ is the characteristic function of B_i . Since

$$(a_i + b, x) = (a_i + b, f(a_i) + c_i) = (a_i, f(a_i)) + (b, c_i) \in L_{b,c_i} \quad (\forall b, \forall i),$$

we have $(s, x) \in \mathcal{P}_{\mathcal{M}_x(s)}$. Note also

$$\sum_{s \in A+B} \mathcal{M}_x(s) = \sum_{i=1}^\tau \sum_{s \in A+B} \chi_{B_i}(s) = \tau|B|.$$

Let $M \triangleq \frac{\tau|B|}{2|A+B|}$. Then

$$\sum_{s \in A+B: \mathcal{M}_x(s) < M} \mathcal{M}_x(s) < |A+B|M = \frac{\tau|B|}{2},$$

and hence

$$\sum_{s \in A+B: \mathcal{M}_x(s) \geq M} \mathcal{M}_x(s) \geq \frac{\tau|B|}{2}.$$

Dyadically decompose this sum, so that

$$(8) \quad \sum_j X_j(x) \gg \tau|B|,$$

where

$$\begin{aligned} X_j(x) &\triangleq \sum_{s: M2^j \leq \mathcal{M}_x(s) < M2^{j+1}} \mathcal{M}_x(s), \\ Y_j(x) &\triangleq \left| \left\{ s \in A+B : M2^j \leq \mathcal{M}_x(s) < M2^{j+1} \right\} \right|. \end{aligned}$$

By (7),

$$\sum_{x: \sigma_{f(A), C}(x) \geq \tau} Y_j(x) \leq |\mathcal{P}_{M2^j}| \ll \frac{|B|^2 |C|^2}{M^3 2^{3j}}.$$

Noting that $X_j(x) \approx Y_j(x) M 2^j$, thus

$$\sum_{x: \sigma_{f(A), C}(x) \geq \tau} X_j(x) \ll \frac{|B|^2 |C|^2}{M^2 2^{2j}},$$

which followed by first summing all j 's, then applying (8), gives

$$\tau |B| \cdot |\{x : \sigma_{f(A), C}(x) \geq \tau\}| \ll \frac{|B|^2 |C|^2}{M^2}.$$

Equivalently,

$$|\{x : \sigma_{f(A), C}(x) \geq \tau\}| \ll \frac{|A + B|^2 |C|^2}{|B| \tau^3}.$$

This finishes the proof of (5).

In the same way one can prove (6). We only sketch the proof as follows and leave the details to the interested readers: Suppose $\sigma_{A, B}(y) \geq \tau$. There exist τ distinct elements $\{a_i\}_{i=1}^\tau$ from A , τ distinct elements $\{b_i\}_{i=1}^\tau$ from B , such that $y = a_i + b_i$. Then we define $C_i \triangleq f(a_i) + C$ and $\mathcal{M}_y(s) \triangleq \sum_{i=1}^\tau \chi_{C_i}(s)$, and as before, $(y, s) \in \mathcal{P}_{\mathcal{M}_y(s)}$. In precisely the same way as the proof of (5), one can prove that

$$\begin{aligned} \sum_{s: f(A) + C: \mathcal{M}_y(s) \geq M} \mathcal{M}_y(s) &\geq \frac{\tau |C|}{2}, \\ \sum_{y: \sigma_{A, B}(y) \geq \tau} Y_j(y) &\leq |\mathcal{P}_{M2^j}| \ll \frac{|B|^2 |C|^2}{M^3 2^{3j}}, \\ \sum_{y: \sigma_{A, B}(y) \geq \tau} X_j(y) &\ll \frac{|B|^2 |C|^2}{M^2 2^{2j}}, \\ \tau |C| \cdot |\{y : \sigma_{A, B}(y) \geq \tau\}| &\ll \frac{|B|^2 |C|^2}{M^2}, \\ |\{y : \sigma_{A, B}(y) \geq \tau\}| &\ll \frac{|f(A) + C|^2 |B|^2}{|C| \tau^3}, \end{aligned}$$

where $M \triangleq \frac{\tau |C|}{2|f(A) + C|}$, $X_j(y) \triangleq \sum_{s: M2^j \leq \mathcal{M}_y(s) < M2^{j+1}} \mathcal{M}_y(s)$, $Y_j(y) \triangleq |\{s \in f(A) + C : M2^j \leq \mathcal{M}_y(s) < M2^{j+1}\}|$. This finishes the whole proof. \square

Corollary 3.3. *Let f be a continuous, strictly convex or concave function on the reals, and $A, C, F \subset \mathbb{R}$ be finite sets such that $|A| \approx |C| \ll |F|$. Then*

$$(9) \quad E(A, A) \ll E_{1.5}(A)^{2/3} |f(A) + C|^{2/3} |A|^{1/3},$$

$$(10) \quad E(A, F) \ll |f(A) + C| |F|^{3/2},$$

$$(11) \quad E_3(A) \ll |f(A) + C|^2 |A| \log |A|,$$

$$(12) \quad E(f(A), f(A)) \ll E_{1.5}(f(A))^{2/3} |A + C|^{2/3} |A|^{1/3},$$

$$(13) \quad E(f(A), F) \ll |A + C| |F|^{3/2},$$

$$(14) \quad E_3(f(A)) \ll |A + C|^2 |A| \log |A|.$$

Proof. Let $\Delta > 0$ be an arbitrary real number. First decomposing $E(A)$, then applying Lemma 3.2 with $B = -A$, gives

$$\begin{aligned} E(A, A) &= \sum_{s: \delta_A(s) < \Delta} \delta_A(s)^2 + \sum_{j=0}^{\lfloor \log |A| \rfloor} \sum_{s: 2^j \Delta \leq \delta_A(s) < 2^{j+1} \Delta} \delta_A(s)^2 \\ &\ll \sqrt{\Delta} \cdot E_{1.5}(A) + \sum_{j=0}^{\lfloor \log |A| \rfloor} \frac{|f(A) + C|^2 |A|}{2^{3j} \Delta^{3j}} \cdot 2^{2j} \Delta^{2j} \\ &\ll \sqrt{\Delta} \cdot E_{1.5}(A) + \frac{|f(A) + C|^2 |A|}{\Delta} \quad (\triangleq \Psi(\Delta)). \end{aligned}$$

Thus $E(A) \ll \min_{\Delta > 0} \Psi(\Delta) \approx E_{1.5}(A)^{2/3} |f(A) + C|^{2/3} |A|^{1/3}$, which proves (9). Similarly, applying Lemma 3.2 with $B = -F$, gives

$$\begin{aligned} E(A, F) &= \sum_{s: \delta_{A,F}(s) < \Delta} \delta_{A,F}(s)^2 + \sum_{j=0}^{\lfloor \log |A| \rfloor} \sum_{s: 2^j \Delta \leq \delta_{A,F}(s) < 2^{j+1} \Delta} \delta_{A,F}(s)^2 \\ &\ll \Delta \cdot E_1(A, F) + \sum_{j=0}^{\lfloor \log |A| \rfloor} \frac{|f(A) + C|^2 |F|^2}{|C| 2^{3j} \Delta^{3j}} \cdot 2^{2j} \Delta^{2j} \\ &\ll \Delta |A| |F| + \frac{|f(A) + C|^2 |F|^2}{|C| \Delta} \quad (\triangleq \Phi(\Delta)). \end{aligned}$$

Thus $E(A, F) \ll \min_{\Delta > 0} \Phi(\Delta) \approx |f(A) + C| |F|^{1.5}$, which proves (10). Once again applying Lemma 3.2 with $B = -A$, gives

$$\begin{aligned} E_3(A) &= \sum_{j=0}^{\lfloor \log |A| \rfloor} \sum_{s: 2^j \leq \delta_A(s) < 2^{j+1}} \delta_A(s)^3 \\ &\ll \sum_{j=0}^{\lfloor \log |A| \rfloor} \frac{|f(A) + C|^2 |A|}{2^{3j} \Delta^{3j}} \cdot 2^{3j} \Delta^{3j} \approx |f(A) + C|^2 |A| \log |A|, \end{aligned}$$

which proves (11). (12)~(14) can be established by the same way. This concludes the whole proof. \square

4. PROOFS OF THE MAIN RESULTS

4.1. Proof of Theorem 1.1. First, apply Hölder's inequality as follows to bound $E_{1.5}(A)$ from below:

$$|A|^6 = \left(\sum_{s \in A-A} \delta_A(s) \right)^3 \leq \left(\sum_{s \in A-A} \delta_A(s)^{1.5} \right)^2 |A - A| = E_{1.5}(A)^2 |A - A|.$$

Therefore, using the above bound and Lemma 2.1 with $B = -A$ gives

$$\frac{|A|^8}{|A - A|} \leq E_{1.5}(A)^2 |A|^2 \leq E_3(A) E(A, A - A).$$

Finally, apply (11), and (10) with $F = A - A$, to conclude that

$$\frac{|A|^8}{|A - A|} \ll |f(A) + C|^3 |A - A|^{3/2} |A| \log |A|,$$

and hence

$$|f(A) + C|^6 |A - A|^5 \gg \frac{|A|^{14}}{(\log |A|)^2},$$

as required.

4.2. Proof of Theorem 1.2. Using the standard Cauchy-Schwarz bound on the additive energy, and then (9), we see that

$$\begin{aligned} \frac{|A|^{12}}{|A + A|^3} &\leq E(A, A)^3 \\ &\ll E_{1.5}(A)^2 |f(A) + C|^2 |A| \\ &= \left(\frac{|f(A) + C|^2}{|A|} \right) E_{1.5}(A)^2 |A|^2. \end{aligned}$$

Next, apply Lemma 2.1, with $B = A$, to get

$$\frac{|A|^{12}}{|A + A|^3} \ll \left(\frac{|f(A) + C|^2}{|A|} \right) E_3(A) E(A, A + A),$$

and then apply (11), and (10) with $F = A + A$, to get

$$\frac{|A|^{12}}{|A + A|^3} \ll \frac{|f(A) + C|^2}{|A|} |f(A) + C|^3 |A + A|^{3/2} |A| \log |A|,$$

which, after rearranging, gives

$$|f(A) + C|^{10} |A + A|^9 \gg \frac{|A|^{24}}{(\log |A|)^2}.$$

4.3. Proof of Theorem 1.3. Observe that the Cauchy-Schwarz inequality applied twice tells us that

$$\frac{|A|^{24}}{|A + f(A)|^6} \leq E(A, f(A))^6 \leq E(A, A)^3 E(f(A), f(A))^3,$$

so that after applying (9) and (12), with either $C = A$ or $C = f(A)$,

$$\begin{aligned} \frac{|A|^{26}}{|A + f(A)|^6} &\leq |A|^2 \cdot E_{1.5}(A)^2 |A + f(A)|^2 |A| \cdot E_{1.5}(f(A))^2 |A + f(A)|^2 |A| \\ &= (E_{1.5}(A)^2 |f(A)|^2) \cdot (E_{1.5}(f(A))^2 |A|^2) \cdot |A + f(A)|^4 \\ &\leq E_3(A) E_3(f(A)) E(A, A + f(A)) E(f(A), A + f(A)) |A + f(A)|^4, \end{aligned}$$

where the last inequality is a consequence of two applications of Lemma 2.1. Next apply (11) and (14), again with either $C = A$ or $C = f(A)$, to get

$$\frac{|A|^{26}}{|A + f(A)|^6} \leq |A + f(A)|^8 |A|^2 (\log |A|)^2 E(A, A + f(A)) E(f(A), A + f(A)).$$

Finally, apply (10) and (13), still with either $C = A$ or $C = f(A)$, so that

$$\frac{|A|^{26}}{|A + f(A)|^6} \leq |A + f(A)|^{13} |A|^2 (\log |A|)^2.$$

Then, after rearranging, we get

$$|A + f(A)| \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}}.$$

Acknowledgements. This first listed author was supported by the NSF of China (11001174). The second listed author would like to thank Misha Rudnev for many helpful conversations.

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